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THE BIOSTATISTICS OF SENILITY

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THE decay of vitality with age is a biological fact most recognize in themselves and none fail to recognize in others; but the biometry of the subject is a difficult undertaking. There is indeed an immense literature, ranging from the moral and psychological observations of Cicero which, of course, edified us in school days to actuarial discussions on the graduation of rates of mortality at advanced ages. To a writer with a predilection for arithmetic the last named studies are more attractive than even Cicero, but he soon discovers that the actuarial treatment of mortality at advanced ages is a matter of art rather than science; partly because the data are either scanty or unreliable, partly because the financial interest of the subject is negligible. In a recent paper Huber has summarized the almost bewildering variety of expedients adopted by official statisticians in different countries to bring life tables to a more or less reasonable end.

To the biostatistician who uses arithmetic, or even algebra, as a tool and not an end in itself, the biological basis of senescence must always be a primary object of curiosity. One does not reach biological truth merely by doing sums. Doing sums, however, is one of the best ways of differentiating biological truth from error.

Fifteen years ago, Raymond Pearl in a course of lectures on the Biology of Death gave a critical survey of relevant biological literature. One conclusion he reached was that, until more accurate observations had been assembled, theorizing was only an agreeable intellectual amusement. He and his colleagues settled down to the exact comparison of mortality rates in different biological types and, Pearl and Miner (1935) published what, in our opinion, is the best contribution yet made to the subject. We will quote a concluding paragraph. "In the meantime it seems to us that the crying need is for more observational data, carefully and critically collected for different species of animals and plants, that will follow through the life history from birth to death of each individual in a cohort of statistically respectable magnitude. From such data sound and biologically meaningful life tables can be constructed. Work of this character, laborious as it is, is likely to be more fruitful of real knowledge than the construction of any mathematical 'law' of mortality, however ingenious." (Pearl and Miner, *op. cit.* 1935).

This finding confirms the conclusion one of us had drawn in 1928 from much less extensive data, and we have no doubt that the plan proposed is the right strategy. But experience of experimental epidemiology, which is virtually an application of identical scientific principles to the study of disease as a biological mass phenomenon, leads us to believe that few have Charles Darwin's iron self-control and can refrain from speculation in advance of data.

Recently Dr. E. J. Gumbel published a monograph (1937), *La durée extrême de la vie humaine,* which, for several reasons, merits study. As is well known, Gumbel has made important contributions to the study of the sampling-frequency distribution of the greatest or least values of a population, a problem first raised by Francis Galton which other mathematicians, notably Fisher and Tippett (1928), have investigated. Gumbel applied his methods to the particular case of survivorship and in the course of his work reached a result which he termed the paradox of the greatest age. We propose first of all to resume briefly the mathematical argument. Gumbel uses the following notation:—

$l(x) = \int_x^\infty \theta(x)\,dx$ where $\theta(x)$ is the probability that a new-born child will die between the ages x and $x + dx$. ξ is the value of x for which $\theta(x)$ is a maximum, and termed the normal age. $\mathrm{e}(x)$ is the expectation of life at age x.

The probability, $W_N(x)$ that the oldest of N decedents is not greater than x is $W_N(x) = \left(1 - l(x)\right)^N$
and the density of probability $w_N(x)$ is

$$w_N(x) = N\left(1 - l(x)\right)^{N-1}\theta(x) \qquad (1)$$

The modal age of (1) is denoted by $\tilde{\omega}$ and

$$t = \frac{x - \xi}{e\,(\xi)}\,,\ \tau = \frac{\tilde{\omega} - \xi}{e\,(\xi)}$$

Now suppose we have two life tables for which $\xi_1 > \xi_2$ but $e_1\,(\xi_1) < e_2\,(\xi_2)$

Then Gumbel reaches the conclusion that

$$\tilde{\omega}_1 \overset{<}{\underset{>}{=}} \tilde{\omega}_2 \qquad (2)$$

if

$$\tau \overset{>}{\underset{<}{=}} (\xi_1 - \xi_2) \Big/ \Big\{ e_2\,(\xi_2) - e_1\,(\xi_1) \Big\} \qquad (3)$$

so that for the more favorable table (i.e. the table for which the normal age is older) the most probable value of the "oldest age" of a sufficiently large group[1] is less than in the unfavorable table. For instance the normal age of white males 1901 U. S. A. experience was 75.05 years, of colored males 56.92 years. The "oldest ages" deduced by Gumbel were 113 and 121 years. The corresponding observed values were 119 and 128 years.

We shall summarize the mathematical argument. Differentiating (1) to obtain a maximum we have

$$\frac{N-1}{1 - l\,(\tilde{\omega})}\,\theta\,(\tilde{\omega}) + \frac{\theta'(\tilde{\omega})}{\theta\,(\tilde{\omega})} = 0 \qquad (4)$$

A first approximation is reached by assuming N large in comparison with unity and unity large in comparison with $l\,(\tilde{\omega})$, so approximately

$$N = -\frac{\theta'(\tilde{\omega})}{\theta^2(\tilde{\omega})} \qquad (5)$$

Now suppose that the force of mortality at the oldest age may be represented by:—

$$\mu\,(\tilde{\omega}) = -\frac{\theta'(\tilde{\omega})}{\theta\,(\tilde{\omega})} \qquad (6)$$

it follows that

$$l\,(\tilde{\omega}) = \frac{1}{N} \qquad (7)$$

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[1] This quantity is called by Gumbel the *"dernier age"*. We translate this "oldest age", the inverted commas indicating the technical use of the expression.

If this holds it may be shown that

$$\tilde{\omega} = \xi + \mathrm{e}\ (\xi)\ \mathrm{F}\ [\log N\ l\ (\xi)] \tag{8}$$

which implies that for $x \geqq \xi$

$$l\ (x) = l\ (\xi)\ \exp\left[-f\left(\frac{x-\xi}{\mathrm{e}\ (\xi)}\right)\right]$$

where $f\left(\frac{x-\xi}{\mathrm{e}\ (\xi)}\right)$ is a steadily increasing function of x with at least two differential coefficients, and F is the inverse function of f. Equation (5) is clearly reasonable. With respect to (6) we note that, by definition,

$$\mu\ (x) = -\frac{l'(x)}{l\ (x)}$$

and that

$$\lim_{x\to\infty} \frac{l'(x)}{l\ (x)} = -\lim_{x\to\infty} \frac{\theta\ (x)}{l\ (x)} = \lim_{x\to\infty} \frac{\theta'(x)}{\theta\ (x)}$$

If the limiting values of $\theta'(x)$ and $\theta\ (x)$ were both finite and not zero, (6) would be ultimately exact and (7) must hold. But the limits of both $\theta'(x)$ and $\theta\ (x)$ as x tends to infinity are zero.

But it is true that

$$\lim_{x\to\infty} \frac{\mu\ (\tilde{\omega})}{\theta'(\tilde{\omega})/\theta\ (\tilde{\omega})} = 1$$

Also (5) holds and

$$N = \frac{1}{l\ (\tilde{\omega})}\left(1 - \frac{\mu'(\tilde{\omega})}{\mu^2(\tilde{\omega})}\right) \tag{9}$$

follows from the definition of $\mu\ (x)$ and the relation :—

$$\theta'(x) = l\ (x)\ \ (\mu'(x) - \mu^2(x))$$

We think it is difficult to conceive a 'law' of mortality such that $\frac{\mu'(x)}{\mu^2(x)}$ will not tend to zero as x tends to infinity. Hence, we think, Gumbel's argument is sound and that the 'paradoxical' result stated above follows. We may illustrate the result easily on so simple a 'law' as that of Gompertz. Here $l\ (x) = k\ g^{c^x}$ where c is greater and g less than unity. It may easily be shown (see Gumbel 1937, p. 15) that the force of mortality at the normal age is equal to the natural logarithm of c. Now suppose we have two tables both following Gompertz's law from any age we please in relatively early life. Take, for instance, The Manchester Township Life Table (Males) 1881-90 and English Life Table No. 10 (Males). The former is a very unfavorable, the latter a very favorable table. For the former the normal age is between 50 and 51 and the expectation of life at that age between 12.8 and 12.4 years. For the latter the normal age is between 74 and 75 and the expectation of life at that age about 6.8 years. Now the force of mortality at 74 by English Life Table No. 10 is greater than the force of mortality at 50 by the Manchester Township Table (the approximate values are 0.0865 and 0.0477). If, therefore, both tables followed Gompertz's law, the relations being:—

For English Life Table No. 10 $_1l\ (x) = k_1\ g_1{}^{c_1^x}$

For Manchester Township Table $_2l\ (x) = k_2\ g_2{}^{c_2^x}$

it is clear that $c_1 > c_2$

If we suppose Gompertz's law to apply exactly to both tables from, say, age 60, then as from 100,000 entrants, the survivors at 60 are respectively 63,620 and 14,945, or $_1l\ (o) > {}_2l\ (o)$. Therefore $k_1\ g_1 > k_2\ g_2$

Write $$\left\{ {}_1l\ (x) \ / \ {}_2l\ (x) \right\} = r_x$$

Then

$$\log r_x = \log k_1 - \log k_2 + c_1{}^x \log g_1 - c_2{}^x \log g_2 \qquad (10)$$

Putting $x = o$, $\log r_0 > o$ because $k_1\ g_1 > k_2\ g_2$

But; since $c_1 > c_2 > 1$ and both $\log g_1$ and $\log g_2$ are negative, we can by increasing x sufficiently make $\log r_x$ negative.

Hence the table giving the large proportion of survivors at a relatively early age must give the smaller proportion at some very advanced age and conversely. This illustrates Gumbel's reasoning.

The whole argument, of course, depends on the assumption that the two populations, or tables, follow the same law — whatever the law may be. That it is not necessarily true that ultimate survivorship will be less favorable under a good than under a bad table may easily be shown. Pearl (1928) published life tables for two genetically distinct races of *Drosophila,* the long-lived wild type and the short-lived vestigial type. Line 107 of the former (males) had a normal age of 51 days (approximately) and an expectation of life at that age of 10 days. Vestigial males had a normal age of 11 days and a normal expectation of 7.9

days. A straight comparison will create no paradox for the normal expectation of the vestigial flies is less than that of the wild flies. But we can easily choose a mixture of the two such that, for the mixture, the expectation of life at the (younger) normal age for the mixture will be greater than the expectation of life at 51 days for the wild flies. If, for instance, we form a life table by taking 1/10th of the l_x for wild flies and 9/10ths for the l_x for vestigial flies, the normal age will still be approximately 11 days but the expectation of life at that age will be 10.76 days as against 10 days in the wild type. But biologically speaking the presence of the short-lived flies is irrelevant; ultimate survivorship depends wholly upon the wild line. This does not, of course, impugn in any way Gumbel's reasoning; *ex hypothesi*, the 'law' of mortality in the mixture could not in general be of the same form as that of the wild type.

This may be shown as follows. Let us assume that the laws of mortality were the same for the wild type and the mixture, and of the general form postulated by Gumbel. Then we have:—

Wild type $$l_1(x) = l_1(\xi_1)\exp\left[-f\left(\frac{x-\xi_1}{e(\xi_1)}\right)\right] \qquad (11)$$

Mixture $$l_2(x) = l_2(\xi_2)\exp\left[-f\left(\frac{x-\xi_2}{e(\xi_2)}\right)\right] \qquad (12)$$

Here

$$\xi_1 = 51,\ e(\xi_1) = 10,\ \xi_2 = 11,\ e(\xi_2) = 10.76$$

thus

$$\xi_1 > \xi_2 \text{ and } e(\xi_1) < e(\xi_2)$$

Since ultimate survivorship depends wholly upon the wild line we must assume that

$$\left\{l_2(x) / l_1(x)\right\} \to 0.1 \text{ as } x \to \infty .$$

Let

$$x = \xi_1 + a = \xi_2 + (a + \xi_1 - \xi_2)$$

then

$$l_1(x) = l_1(\xi_1)\exp\left[-f\left(\frac{a}{e(\xi_1)}\right)\right]$$

$$l_2(x) = l_2(\xi_2)\exp\left[-f\left(\frac{\xi_1-\xi_2+a}{e(\xi_2)}\right)\right]$$

Hence

$$\log l_2(x) - \log l_1(x) =$$
$$\log l_2(\xi_2) - \log l_1(\xi_1) + f\left(\frac{a}{\mathbf{e}(\xi_1)}\right) - f\left(\frac{\xi_1 - \xi_2 + a}{\mathbf{e}(\xi_2)}\right) \quad (13)$$

Now ultimately $\xi_1 - \xi_2$ and $\log l_1(\xi_1) - \log l_2(\xi_2)$ are both very small compared with a, and since $\mathbf{e}(\xi_2) > \mathbf{e}(\xi_1)$ the right hand side of this expression will ultimately become positive and large. For instance with Lexis' hypothesis $f(u) \sim \left(\frac{u^2}{\pi} + \log 2u\right)$ and

$$\log l_2(x) - \log l_1(x) = \frac{a^2}{\pi}\left[\frac{1}{\mathbf{e}^2(\xi_1)} - \frac{1}{\mathbf{e}^2(\xi_2)}\right] \text{ approximately,}$$

that is the ratio of $l_2(x)$ to $l_1(x)$ becomes very large. But this contradicts the hypothesis that $[l_2(x) / l_1(x)] \to 0.1$. Hence the form of the function f cannot be the same in both cases.

We are now brought to an important and highly relevant consideration. It has long been suspected that a genetic factor of longevity exists; the work of Pearl and his collaborators has greatly strengthened the case for believing that capacity to live long is heritable. If this be true it follows that with advancing age the quality of the population changes, in fact the analogy between a mixture of wild and vestigial flies and the human population may be an exact analogy.

If we further suppose that, under an unfavorable environment, the wild flies are more resistant to the factors causing death before old age has begun, we could understand how, in a 'bad' table, rates of mortality at relative advanced ages might be more favorable than in a 'good' table. But it is not easy to understand how the age attained on the average by one in a vast number of entrants to life would be affected, provided that the initial proportion of wild flies in the population is the same under the two environments. We are, therefore, led to consider whether there is any good reason to believe that a uniform law of mortality does apply to the facts of human experience. The merits and demerits of the Gompertz or Makeham-Gompertz formulae have been discussed many times (see e.g. Greenwood, 1928). Gumbel has given attention to Lexis' system, which has also been warmly praised by Freudenberg. This is less familiar than the Makeham-Gompertz or Gompertz formulae and we shall describe it in some detail.

Lexis's theory of mortality was first published in 1877 (Lexis, 1877 pp. 42 *et seq.*) and again stated with little change in 1903 (see Lexis, 1903 pp. 111 *et seq.*). He, following to some extent Quetelet but, with a nuance of Platonism, thought of a typical or normal age of man to the attainment of which the nature of things was striving. Were there no disturbing factors, the life table distribution of deaths in a generation would be a normal curve, the origin of which would be the normal age. But in real life there would be disturbing factors. In the first place some of those born would be constitutionally unfit for long survival; in the second place unfavorable environmental factors would lead to the premature death of some fitted to reach the normal age. Lexis illustrated his theory by an analogy. He imagined the case of a man throwing balls at a mark who, if undisturbed, would distribute these balls on either side of the mark in a Gaussian distribution, but who was actually disturbed in two ways. Firstly, some of the balls he picked up seemed too light or too heavy; these he dropped at his feet. Secondly a mischievous person ran along the skittle alley, put out his hand, caught and dropped some of the balls in flight, confining his attention to the nearer side of the alley, and not interfering with over-pitched balls. Under these conditions the over-pitched balls will be distributed in a Gaussian way so that one will have from the normal age the half of a normal curve, while, prior to that age, the distribution would have an early maximum and then approach slowly to the value at the normal age. Lexis tested this hypothesis on a large number of life tables. His method was, starting with the tabular values of d_x to fix the normal age (i.e. the age for which d_x was maximal) and then, usually on a range of about 8 years, to deduce the value of the standard deviation. E.g. if the normal age were 72, the number of deaths between 72 and 80 were 100, and the whole number of deaths beyond 72 were 200, one would enter Sheppard's table for $\frac{1}{2}(1 + a) = 0.75$ obtain the value of $8/\sigma$ and so deduce σ. On many tables the agreement between observation and expectation seemed to Lexis satisfactory.

It may be urged that the statistical method is rather crude and that in fact no precise test of agreement between observation and theory was applied. Lexis, however, belonged to an order of intelligence much above the level at which schoolmasterly criticisms are appropriate. So far as the question of method is concerned, he did not need to be told that an arithmetically better fit might easily be obtained for a given set of data by using more of the data. He only used a small range because he knew that the further one went in a published life table the more sophisticated by interpolation, graduation, etc., were the values printed in it and he wished to base his argument on a range over which the data were least doctored.

So far as the question of fit is concerned, he was quite aware that so simple a law as he proposed could hardly account for the whole of the facts, and was, therefore, content to reach approximate results. Freudenberg (1934), using a more refined process of deducing the standard deviation and working with more modern data also reached apparently good concordance of d_x's beyond the normal age with those required by the hypothesis of Lexis. But to say this is not to say that the hypothesis of Lexis is free from difficulty. In the first place it is to be noted that the normal age in Lexis's sense has moved further on in life. In the terms of Lexis's parable, the skittle player has fixed his eye on a mark further down the alley. Freudenberg (*op. cit.*, p. 383) using German Life Tables from 1871-1881 to 1924-1926 has an increase from 70.1 years to 75.4 years for males (σ decreased from 9.53 to 8.12) and from 72.2 to 76.2 years for females (σ decreased from 8.30 to 8.20).

We have used four tables based on English experience, viz. E. L. 7 and 10 and the 'ultimate' tables based on annuitant experience by Elderton and Oakley (1924). In the national tables the range of normal ages is much smaller (as one would expect) but while for E. L. 7 (the mortality rates prevailing 30 years ago) the normal age is 79.8 years it is 85.5 years for the very select annuitants. It is at least clear that our hypothetical skittle player shifts his mark. This is rather difficult to reconcile with the original hypothesis. The argument was that, ideally, we should have a normal frequency distribution of the variate d_x and that practically we did *not* have such a distribution, partly because the data themselves were distorted by manipulation but partly — and in greater part — by the intrusion of deaths, the juvenile and premature deaths, which were strictly speaking accidental impurities. With the reduction of these by environmental betterment the pure idea should emerge more distinctly. But the normal age could hardly move unless we suppose that the mark is changed. If, owing to hygienic betterment or selection, the *quality* of young lives is improved — in terms of the analogy the box of balls used by the thrower contains fewer too light or too heavy balls — one would perhaps expect the scatter to diminish, but hardly the position of the target.

In spite of our respect for Lexis as a statistician, we cannot discern in his treatment of the mortality table any biological superiority to that of, for instance, Karl Pearson, who dissected the table into a series of components, a method adopted in principle by Arne Fisher. The Lexian method has an advantage over the Pearsonian method in so far as fewer constants are needed (see Pearl, 1922, pp. 94-100) but not, so far as appears, any intrinsic merit. The analogy of the skittle player is only an analogy, without biological justification.

Possibly such relative advantage over the method of Gompertz as Freudenberg found means no more than that the rate of mortality increases with age more slowly than the 'law' of Gompertz requires. Gumbel has pointed out that it must follow algebraically that when the

TABLE 1

English Life Table No. 10 (Females)

AGE	'OBSERVED' d_x	CALCULATED (LEXIAN METHOD)
80	2949	2860.5
81	2834	2795.9
82	2684	2709.4
83	2505	2550.7
84	2303	2333.6
85	2080	2116.8
86	1843	1864.9
87	1599	1597.0
88	1347	1355.9
89	1114	1117.9
90	905	896.1
91	720	712.2
92	561	549.5
93	426.9	412.3
94	316.9	306.7
95	229.0	221.4
96	160.8	156.4
97	109.5	107.4
98	72.2	73.2
99	45.9	48.4
100	28.1	31.1
101	16.6	19.8
102	9.3	12.3
103	5.0	7.4
104	2.6	4.4
105	1.2	2.8
106	0.6	3.1

TABLE 2

Graduations of various tables by Gompertz's method (G) and Lexis's method (L). Tabular values (d_x's) expressed as percentages of the graduated values

AGE	MANCHESTER CITY 1881-90		MANCHESTER CITY 1881-90		E.L. NO. 7		E.L. NO. 7		E.L. NO. 10		E.L. NO. 10	
	M.G.	M.L.	F.G.	F.L.	M.G.	M.L.	F.G.	F.L.	M.G.	M.L.	F.G.	F.L.
80	94.2	102.4	90.7	101.2	82.8	105.3	80.9	104.1	91.2	102.8	87.4	103.1
81	96.1	100.0	93.2	100.5	85.0	100.6	84.0	100.6	93.7	100.7	90.4	101.4
82	96.9	99.0	95.6	99.9	87.7	97.2	87.2	99.0	95.9	98.6	93.2	99.1
83	98.7	98.2	97.4	99.4	91.0	96.7	90.7	97.4	97.8	98.5	95.8	98.2
84	99.9	98.6	98.9	99.0	94.7	96.3	94.4	97.5	99.4	98.3	98.2	98.7
85	100.7	97.8	101.0	99.6	98.4	97.9	98.0	98.2	100.8	98.5	100.2	98.3
86	101.2	98.6	101.2	98.9	101.7	99.2	101.1	99.7	101.7	98.4	101.8	98.8
87	101.3	98.1	101.3	98.3	104.4	100.0	103.3	99.8	102.2	99.7	102.7	100.1
88	101.6	99.4	102.7	99.4	106.7	102.3	105.0	100.6	101.7	99.9	102.4	99.3
89	102.6	100.6	102.4	99.2	108.7	103.6	106.1	101.5	101.2	100.3	101.9	99.6
90	103.4	102.7	103.6	100.5	109.5	105.6	106.5	102.3	100.8	100.3	101.6	101.0
91	99.5	99.0	100.7	98.7	108.8	105.5	106.2	103.2	100.5	102.0	101.2	101.1
92	102.2	102.9	101.5	99.7	106.1	103.2	105.1	102.2	100.2	102.6	100.8	102.1
93	103.1	106.4	102.4	101.6	102.0	100.7	103.2	101.4	99.8	102.3	100.5	103.5
94	89.6	92.3	99.5	100.0	97.4	96.3	101.1	100.3	99.7	103.0	100.2	103.3
95	111.1	116.3	99.3	100.7	93.3	93.2	98.8	98.8	99.4	102.2	99.9	103.4
96	69.0	69.0	97.1	99.0	89.9	89.6	96.9	98.0	99.5	100.7	99.6	102.8
97	100.0	105.3	94.6	97.2	89.0	87.4	95.6	96.0	99.2	97.3	99.4	102.0
98	83.3	83.3	94.3	100.0	90.3	87.9	94.9	94.8	99.1	94.1	99.2	98.6
99	142.9	142.9	108.1	114.3	93.8	88.0	95.6	94.2	100.0	89.8	98.9	94.8
100	100.0	90.9	80.0	83.3	100.4	90.9	96.9	93.2	100.0	83.3	98.6	90.4
101	....	....	117.6	125.0	109.0	93.4	99.9	93.2	100.0	74.1	99.4	83.8
102	....	....	90.9	90.9	114.5	89.6	100.9	88.9	100.0	67.7	98.9	75.6
103	....	....	142.9	142.9	117.6	84.0	106.6	88.0	100.0	55.6	98.0	67.6
104	....	....	83.3	83.3	97.6	60.6	105.5	79.8	....	....	....	....
105	....	....	....	....	60.1	27.8	105.8	72.4	....	....	....	....

same data are graduated by the two methods, the Lexian values of l_x will first be less and then greater than the corresponding Gompertzian values. At this point some arithmetical illustrations may be helpful. We took first[2] English Life Table No. 10 (females) treated the d_x column for age 80 onwards as the tail of a normal curve and obtained

TABLE 3

Ultimate table of annuitant experience (Elderton and Oakley). Graduations by Gompertz's method (G) and Lexis's method (L). Tabular values (d_x's) expressed as percentages of the graduated values

	M.G.	M.L.	F.G.	F.L.
80	99.9	105.6	....	....
81	100.0	102.4	....	....
82	100.0	98.8	....	....
83	100.0	97.7	....	....
84	100.0	96.0	....	....
85	100.0	96.5	94.0	104.5
86	100.0	96.2	96.2	100.8
87	100.0	97.2	98.0	99.2
88	100.0	99.2	99.4	97.4
89	100.0	100.2	100.3	97.4
90	100.0	102.7	101.0	97.5
91	100.0	103.8	101.3	97.6
92	100.0	106.1	101.3	99.3
93	100.0	106.5	101.1	100.7
94	100.0	107.2	100.7	101.6
95	100.0	107.6	100.2	103.8
96	100.0	105.4	99.7	105.1
97	100.0	103.3	99.2	105.4
98	100.0	98.3	98.7	106.4
99	100.0	92.4	98.5	105.1
100	100.0	85.5	98.5	104.0
101	100.0	75.9	98.9	100.8
102	99.9	66.4	99.8	95.6
103	100.0	55.3	101.3	90.0
...	....	....	....	....

[2] We are much indebted to our colleagues Dr. Martin and Mr. Cheeseman for carrying out this and much more heavy arithmetical work for us.

the constants by Pearson's method.[8] The result (see Table 1) is, from the graphical point of view, excellent. Indeed grouping the values from age 98 onwards together, the fit, on the basis of the life table numbers is excellent, for $\chi^2 = 9.5$. This would, on the tabular numbers, imply an excellent fit. But the actual number of deaths at ages over 80 from which the table as calculated was 111,746 so that one should multiply the value of χ^2 by almost 4.49 for a strict test. Perhaps, however, such comparisons are unreasonable. It would be naive to expect good fits in the sense of the test, because, *inter alia,* one is using as data not the original material but artificially graduated figures. The fairest comparison of Gompertzian with Lexian results would seem to be to show the respective percentage deviations for a wide range of tables. This is done in Table 2. We think the entries lead to the conclusion that neither method has an overwhelming advantage over the other and that neither has any pretention to be regarded as a natural 'law'. Indeed the only table which does give a decisive superiority to one or other method is the Ultimate Table of the Annuitant Experience, Table 3, for which a Gompertzian graduation does completely reproduce the d_x's from age 80; *not* a very strange result as the table was graduated by joining two Gompertz graduations (see Elderton and Oakley, *op. cit.* p. 40)! What *was* a little surprising was that a 'normal' tail did fit the 'facts' quite well over the range 81-90, with a maximum percentage error of 4.0 and only diverged widely after age 98. We conclude (a conclusion long since reached by actuaries) that none of these 'laws' is anything but a useful interpolation formula. Hence, we think, no arithmetical calculation of the 'last age' has much value.

But one has a certain human curiosity regarding the 'true' law of mortality at old ages. Since the data are, statistically speaking, scanty while the practical importance (from the actuarial side) is trivial, the subject had not been discussed in detail, before Gumbel's work. We can only speculate, but a few conjectures may be allowed. Two sets of facts are suggestive. In the first place, as mentioned above, it has long been held that a constitutional factor is of primary importance to the attainment of great length of days; the earlier literature is succinctly discussed by Rolleston (1922, pp. 32 *et seq.*) and more recently Pearl

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[8] Fitting by Lexis' original method from the normal age (77 years for this table) gives a rather less good result. For all other fittings the Pearsonian method has been used.

and his collaborators (see Pearl and Raenkham, 1932, *et al.*) have made substantial contributions to the case in favour of an heritable element.

In the next place a study of the actual data of mortality at very advanced ages — say at ages over 90 — produces the impression that the increase of mortality rate with age advances at a slackening rate, that nearly all, perhaps all, methods of graduation of the type of Gompertz's formula *over-state* senile mortality. The weights of these two sets of data are very unequal. We do not have much doubt that the importance of an hereditary factor of longevity is proven, but the comparatively scanty 'facts' — in the actuarial sense — of statistical mortality do not lend themselves to firm conclusions.

From the first and more weighty evidence we may reasonably conclude that the "law" of senile mortality, meaning for the present the functional relation between force of mortality and age could not well be the same for all members of a population. It must be different for the genetically long-lived and the genetically short-lived. It is different for annuitants and insured persons. With our present knowledge the rates of the very aged could never be predicted from an extrapolation of mortality rates based on earlier ages, the exposed to risk are not random samples of an entering population but selections, and extrapolations will consequently exaggerate senile mortality. The much less weighty evidence, or impression, derived from the scanty data of statistical records justify us in speculating on the *possibility* that with advancing age the rate of mortality asymptotes to a finite value. At first sight this must seem a preposterous speculation. In a labile, highly specialized metazoan, decay must surely continue. One cannot without absurdity believe that, other things equal, a man of 100 is not more likely to die within a year than a man of 90. But other things are *not* equal. Swift's gloomy parable of the Struldbrugs is not without its practical application. With advancing years the disabilities, forcefully described by a large number of poets whom it is needless to quote, restrict activities. Even the juvenile of 60, if ordinarily intelligent, eschews the violent exercises of the child of 40. Centenarians rarely appear in public. A statistical rate of mortality might show no increase with age, if the demands made on the vital forces diminished *pari passu* with the decay of vigour.

Let us think then of a group of aged persons who have attained the age of x (where x is greater than 90 years) as competitors for prizes, the prizes being awarded to those who survive 1, 2, etc., years. Whoever dies at an age less than $x + 1$ has won no prize, he who dies between $x + 1$ and $x + 2$ has won 1 prize and so forth. Then with a mere change of nomenclature we have the statistical situation considered by Yule and Greenwood in their 1920 paper. If the competitors have equal chances of winning prizes, the graduation of the d_x column should be a simple Poisson series; if their chances are unequal the graduation might follow the simple skew form which was christened the Infinitely Compound Poisson distribution.[4]

The graduation could evidently hold only at extreme ages and involves as a corollary that q_x tends to a finite limit less than unity.

In the form convenient for computation the successive frequencies are given by the terms of :—

$$\left(\frac{c}{c+1}\right)^r \left(1 + \frac{r}{(c+1)} + \frac{r(r+1)}{2!\,(c+1)^2} + \ldots\ldots\right) \quad (14)$$

where $m = \dfrac{r}{c}$ and $\mu_2 = \dfrac{r(c+1)}{c^2}$

If $c \rightarrow \infty$, m remaining finite,

$$\left(\frac{c}{c+1}\right)^{mc} \rightarrow e^{-m}$$

and the expression becomes $e^{-m}\left(1 + m + \dfrac{m^2}{2!} + \ldots\ldots\right)$

the simple Poisson.

The limiting value of q_x *is* $\left(\dfrac{r}{r+m}\right)$

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[4] We are aware that this is an over-simplification. If this form of graduation proves satisfactory, it will, we think, lend support to the hypothesis that our populations are heterogeneous with regard to longevity. But the negative binomial cannot be the true law. The form of the distribution to which the hypothesis tends is not strictly independent of the interval of age taken as a unit. If we fit with an interval of one year and then combine our theoretical frequencies for two-year intervals, the new distribution is not a negative binomial in terms of the new interval. Yet a direct fit of a negative binomial to two-year intervals might be good enough for practical purposes of graduation.

There is another point of interest. We show that this hypothesis ultimately leads to a constant q_x and therefore to a constant force of mortality. But this means that ultimately $l(x)$ approaches the exponential form ae^{-bx}. The reader may therefore enquire whether Gumbel's "paradox" would not hold for two populations for which our hypothesis was correct. This does not follow, for Gumbel's paradox depends on the form of the function f in $f[(x-\xi)/e(\xi)]$ being the same for both populations *from the normal age onwards.*

Thus

$$\frac{d_x}{l_0} = \left(\frac{r}{r+m}\right)^r \frac{r\,(r+1)\ldots\ldots(r+x-1)}{x\,!} \left(\frac{m}{r+m}\right)^x \quad (15)$$

$$\frac{l_x}{l_0} = \left(\frac{r}{r+m}\right)^r \frac{r\,(r+1)\ldots\ldots(r+x-1)}{x\,!} \left(\frac{m}{r+m}\right)^x u_x$$

where

$$u_x = 1 + \frac{r+x}{x+1}\left(\frac{m}{r+m}\right) + \frac{(r+x)\,(r+x+1)}{(x+1)\,(x+2)}\left(\frac{m}{r+m}\right)^2 + \ldots\ldots \quad (16)$$

When $x \rightarrow \infty$

$$u_x \rightarrow 1 + \left(\frac{m}{r+m}\right) + \left(\frac{m}{r+m}\right)^2 + \ldots\ldots\ldots = \left(\frac{r+m}{r}\right),$$

$$\text{or} \quad q_x \rightarrow \left(\frac{r}{r+m}\right)$$

In the simple Poisson

$$d_x = \frac{e^{-\lambda}\,\lambda^x}{x\,!}$$

and by a similar agreement we find

$$q_x \rightarrow 1 \text{ as } x \rightarrow \infty.$$

In a life table the curtate expectation of life would be the m of this notation.

If this is of order 6 years (its value at the maximum of d_x in E. L. 10) and if the limiting value of q_x for x tending to infinity were of order 0.5, c must be of order unity and r approximately equal to 6. In other words, the d_x's would be still increasing. Hence the graduation could not apply to any system of d_x's for which e_x was greater than 2, i.e. it would not apply to ages less than about 90.

The method is so simple that a trial of its practical applicability seemed justified. A first test was made on the recorded deaths at ages 92 years and upwards in the Annual Reports of the Registrar-General for 1921-35. It is hardly necessary to say that to treat these figures as the d_x's of a life table is to be deliberately guilty of the fault of constructing a life table from the deaths in a non-stationary population. But as a rough preliminary test on large numbers the plan seemed worth trying.

As a matter of arithmetical fact the q_x's of this pseudo life table for males agreed closely with the smoothed values of E. L. 10:—

Age	q_x *of pseudo table*	q_x *of E.L. 10*
92	0.321	0.320
93	0.348	0.338
94	0.368	0.357
95	0.367	0.376
96	0.397	0.396
97	0.393	0.417
98	0.420	0.438
99	0.432	0.461
100	0.453	0.481

TABLE 4

1921-35. Deaths. (Data from R-G's Reports)

AGE	MALES									
	Obs.	Compound Poisson								
92	4348	....	....	....	....	....	.....	.....	.....	4202.6
93	3202	....	....	....	....	....	.....	.....	3100.5	3324.4
94	2204	....	....	....	....	....	.....	2118.6	2280.1	2260.7
95	1389	....	....	....	....	....	1366.1	1501.1	1486.2	1453.7
96	953	....	....	....	....	924.1	963.7	952.8	927.4	907.8
97	568	....	....	....	562.8	610.0	602.8	581.3	565.8	556.9
98	369	....	....	363.6	373.0	367.6	361.0	347.5	340.4	337.5
99	220	....	215.4	224.7	222.3	214.5	211.4	205.2	202.9	202.7
100	131	123.0	130.7	129.2	127.6	123.1	122.1	120.1	120.2	121.0
101	63	75.2	73.9	73.4	71.8	70.0	69.9	69.9	70.8	71.8
102	40	42.1	40.8	40.0	39.9	39.5	39.7	40.5	41.5	42.5
103	30	22.9	22.2	22.0	22.0	22.2	22.5	23.4	24.3	25.0
104	10	12.2	12.0	12.0	12.1	12.4	12.7	13.5	14.2	14.7
105	8	6.5	6.5	6.5	6.6	6.9	7.1	7.7	8.3	8.6
106	5	3.4	3.5	3.5	3.6	3.8	4.0	4.4	4.8	5.0
107	1	1.8	1.8	1.9	1.9	2.1	2.2	2.5	2.8	2.9
108	.	.9	1.0	1.0	1.1	1.2	1.2	1.4	1.6	1.7
109	.	.5	.5	.6	.6	.7	.7	.8	.9	1.0
110	1	.5}	.7}	.6}	.7}	.9}	.9}	1.3}	1.3}	1.5}

TABLE 4 (*Continued*)

1921-35. Deaths. (Data from R-G's Reports)

AGE	FEMALES									
	Obs.	Compound Poisson								
92	10063	....	....	....	.....	.....	.....	.....	.....	9332.9
93	7473	....	....	....	.....	.....	.....	.....	6998.5	8065.4
94	5433	....	....	....	.....	.....	.....	5138.3	5852.8	5852.6
95	3878	....	....	....	.....	.....	3689.2	4146.7	4125.9	3976.6
96	2650	....	....	....	.....	2561.6	2861.9	2834.7	2728.0	2610.1
97	1789	....	....	....	1771.2	1918.3	1893.3	1821.2	1744.0	1677.0
98	1238	....	....	1215.5	1249.4	1229.6	1180.4	1132.6	1092.0	1062.0
99	774	....	740.1	778.5	768.8	743.9	713.5	690.4	674.2	665.5
100	426	405.4	462.1	455.8	450.0	436.6	423.1	415.2	412.0	413.8
101	249	272.3	265.0	258.5	256.6	251.6	247.7	247.2	249.9	255.7
102	153	157.4	147.4	144.2	144.1	143.2	143.6	146.1	150.6	157.2
103	80	86.1	80.8	79.7	80.0	80.7	82.7	85.9	90.4	96.2
104	48	45.8	43.8	43.7	44.1	45.2	47.4	50.2	54.0	58.7
105	37	23.9	23.6	23.9	24.1	25.2	27.0	29.3	32.1	35.7
106	13	12.3	12.7	13.0	13.2	14.0	15.3	17.0	19.1	21.7
107	7	6.3	6.8	7.1	7.2	7.7	8.7	9.8	11.3	13.1
108	2	3.2	3.6	3.8	3.9	4.3	4.9	5.7	6.7	7.9
109	1	3.3}	4.1}	4.3}	4.4}	5.1}	6.3}	7.7}	9.5}	11.9}

TABLE 5

Values of χ^2 for deaths (1921-35) observed and calculated by Compound Poisson. (Last two ages grouped.)

MALES	χ^2	n'	P	FEMALES	χ^2	n'	P
From age 92	30.1422	16	.012	From age 92	214.2177	15	<.000000
" " 93	25.2692	15	.032	" " 93	131.8712	14	<.000000
" " 94	22.5117	14	.048	" " 94	79.8604	13	<.000000
" " 95	9.6850	13	.643	" " 95	49.9437	12	.000003
" " 96	10.7766	12	.463	" " 96	24.1411	11	.0073
" " 97	6.4359	11	.776	" " 97	12.9341	10	.166
" " 98	6.4711	10	.691	" " 98	13.8827	9	.085
" " 99	6.9111	9	.546	" " 99	16.1209	8	.024
" " 100	7.8754	8	.345	" " 100	13.2187	7	.040

In the next table (Table 4) we have the results of graduation by Compound Poissons for various starting ages and in the following table (Table 5) the Tests for Goodness of Fit (three degrees of freedom are absorbed in the fitting). The results for males are surprisingly good; for females, bad from the accurate point of view, although graphically not preposterous.

We were encouraged to seek more exact data. The Report on the Mortality Experience of Government Life Annuitants, 1900-1920 (S. O. 1924), contains data of interest. Actually in the tabular matter one has not lives but contracts so that the life upon which two annuities have been purchased will have two exposures and, of course, two deaths. For reasons stated this is preferable for the purposes of the report, but not for our object. It means, or may mean, that the exposed lives are appreciably less numerous than the exposed contracts, and perhaps goes a little way to explain the violent fluctuations of the observed rates of mortality. Take for instance the relatively large experience of female annuitants exclusive of the first year after purchase. From age 93 onwards one has the following figures:—

	Exposed to risk	*Deaths*	q_x
93	830	276	0.33253
94	550	159	0.28909
95	372	115	0.309139
96	247	73	0.295546
97	177	59	0.333333
98	113	37	0.327433
99	78	29	0.371794
100	47	17	0.361170
101	28	11	0.392857
102	17	2	0.117647
103	15	5	0.333333
104	8	5	0.6250
105	3	3	1.00000

These are distressingly unreasonable figures on *any* hypothesis. To graduate them on so simple a hypothesis as that of the 'proneness' rule is a desperate undertaking. Still courage is sometimes virtuous. We began by taking the figures precisely as they stood. We chose 830 as radix and obtained the successive values of l_x by multiplication by the p_x's, just as they were. One had then 830, 554, 394, 272, 192, 128, 86, 54, 33, 29, 19, 7, 0. So the d_x's were 276, 160, 122, 80, 64, 42, 32, 21, 4, 10, 12, 7. Taking these as a frequency distribution for $x = 0, 1, 2,$ etc., the mean is 2.13012, the variance 5.987889. Leading to $c = 0.552164$, and $r = 1.176175$. To the nearest integer the values given are 246, 187, 131, 89, 60, 40, 27, 18, 12, 8, 5 and 9. The goodness of fit is contemptible, as it must be having regard to the wild fluctuations at ages over 100. Even if one clubs together the last four values, P cannot be persuaded to rise above .03. Still the result is less grotesque than one might have expected.

In the following table are shown the 'actual' values, those calculated on the 'proneness' hypothesis, those derived from the smoothed life table based on the experience (*op. cit. p.* 45).

Age	*Actual*	*Proneness*	*Life table*
93	276	246.2	233.2
94	160	186.6	176.8
95	122	130.8	130.7
96	80	89.2	94.2
97	64	60.0	66.1
98	42	40.0	45.2
99	32	26.5	30.9
100	21	17.5	21.1
101	4	11.5	14.3
102	10	7.6	9.2
103	12	5.0	5.3
104	7	9.1	6.0

The values of χ^2 are respectively 27.1 and 28.8, both graduations are equally good (or bad) although the assumptions are completely different. On the proneness view, q_x asymptotes to the ultimate value 0.3557, while the graduated life table makes every value of q_x for x greater than 99 more than this and q_{105} is unity. The inference is, *not* that these data confirm the hypothesis but that they do not exclude it. As by age 100 q_x is already within 0.011 of the ultimate value, it would be sufficiently exact to say that from age 100 the population decreases in geometrical progression with a common ratio of 0.65. On that hypothesis the chance a centenarian woman would have of living to be 106 would be about one in thirteen, as compared with nil (on the official table). But the chance of a woman of 93 attaining this age would be only one in 222. Of course from the practical point of view the fact that the expectation of life at all ages over 100 would be nearly 2.4 years so that the fact that annuities would be independent of age, would deter but few purchasers!

The data for male lives in this report are still scantier and the empirical values of q_x at ages over 90 so erratic that we have not ventured to graduate them.

At this point Mr. E. S. Jones, Actuary to the National Debt Office, was kind enough to supply us with the experience of 240 women and 50 men who attained the age of 90 in 1920-22 and continued under observation until death. In the next table (Table 6) we show observed deaths and graduations by Compound Poissons.

TABLE 6

National Debt Office Data. Deaths (1920-22) observed and calculated by Compound Poisson

	1920-22							
	MALES				FEMALES			
Age	Obs.		Compound Poisson		Obs.		Compound Poisson	
90	11		9.1		46		36.0	
91	12		11.6		45		46.3	
92	7		10.1		29		42.9	
93	5		7.3		34		34.6	
94	7		4.8		25		25.8	
95	2	8 (95–97)	3.0	7.1 (95–97)	21		18.3	
96	4		1.8		11		12.6	
97	2		2.3		11		8.4	
98	50		50		10		5.5	
99					3	8 (99–102)	2.3	9.6 (99–102)
100					4		2.3	
101					..		1.4	
102					1		2.3	
					240		240	

Males: $\chi^2 = 3.2091$, $n' = 4$, $P = .3644$

Females: $\chi^2 = 12.7081$, $n' = 8$, $P = .080$

The fits are again not unreasonable. For males, using 7 groups and therefore 4 degrees of freedom $P = 0.36$. For females, using 10 groups and 7 degrees of freedom $P = 0.08$. The limiting values of q_x are 0.439 for women and 0.544 for men. Some tests of the ultimate mortalities in non-human experience were not unfavorable. Thus for

Hydra (Pearl and Miner, p. 61) one has[5] (graduated values in backets):— 18 (17), 12 (13), 6 (7), 4 (3), 1 (1). For starved *Drosophila* (*op. cit.*, p. 64):— 38 (36), 29 (30), 23 (23), 18 (18), 13 (13), 10 (10), 7 (7), 5 (5), 4 (4), 3 (3), 2 (2), 2 (2), 1 (1) and for the tail 3 (3).

It is hardly necessary to add that such agreements in small samples do no more than show that the method of graduation is not preposterous; they go no way to *prove* that the theory is correct. Take for example the following distribution.

Value	*Frequency*
0	4
1	24
2	24
3	24
4	17
5	4
6	2
7	1
	100

The mean and variance are 2.39 and 2.5779, giving for a Compound Poisson $c = 12.7195$ and $r = 30.3996$. The fit is not unsatisfactory — the theoretical frequencies (to the nearest whole number) are 10, 22, 25, 20, 12, 6.3 and 1. For five degrees of freedom one has a P of more than 0.2. The distribution is of the frequency with which the letter *s* occurs in each of the first hundred lines of *Aeneid* XII. It is true that *s* is a letter frequently doubled so that a Compound Poisson might be expected to give a better fit than a Simple Poisson. It is not true that the success of the experiment throws the least light upon Virgil's method of composition. All that may be claimed as shown by the trials is that if and when numerically adequate data are available, the very simple Compound Poisson formula may justify attention.

Biologically speaking the question whether q_x does tend to an ultimate value far short of unity is interesting. Practically speaking, although we think that the chance of an 'Old Parr' emerging under the favorable modern environment is not less than in Stuart times, it is a chance

[5] For *Hydra* we took deaths in five day sets from age 125-129 days; for starved *Drosophilia* deaths in hours from age 50 hours.

without much actuarial importance. It is, if q_x tends to 0.5, a chance of not much more than $(0.5)^{50}$ that a modern centenarian will beat Parr's record.

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